

# BLACK HOLES IN EINSTEIN-MAXWELL THEORY

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ABSTRACT. We prove different variants of known singularity theorems and statements in the context of Thorne's Hoop conjecture for asymptotically flat Maxwell-Einstein theory. In particular, we show that sufficient concentration of energy implies the existence of a region of finite lifetime, and, consequently, of a horizon. Conformal extensions are a useful tool in the main results.

## 1. INTRODUCTION

A long-standing conjecture in mathematical relativity is the so-called *Hoop Conjecture* due to Kip Thorne [35] stating in deliberately vague terms that *there is a constant  $k > 0$  such that horizons and black holes form if and only if a mass  $m$  is concentrated in a region  $U$  whose hoop circumference  $C_H(U)$  is bounded by  $C < 4\pi M$* . Here the *hoop circumference*  $C_H(U)$  of  $U$  is a diameter-like quantity for open subsets of 3-dimensional Euclidean space generalized by Gibbons [14], [9] to the Birkhoff invariant  $\beta$  where

$$\beta(U) := \min\{\max\{\text{diam}(f^{-1}(\{r\})) | r \in [0, 1]\} | f \in C^1(\partial U), \#(\text{Crit}(f)) = 2\},$$

and  $\partial U$  is assumed to have the topology of a sphere. In our result we will instead work with the Schoen-Yau radius  $\text{Rad}$  to be defined later.

Among the possibilities of defining mass in this context, the 'only if' part of the Gibbons-Thorne conjecture was disproved for ADM mass by Mantoulidis and Schoen [20]. In this article, we will interpret 'mass' as referring not to gravitational mass (which cannot be defined locally) but to the rest mass of additional matter fields. So in this sense, the purely gravitational asymptotically flat black holes due to Li and Yu [18] using earlier work of Christodoulou and Klainerman-Rodnianski disprove the 'only if' part of the conjecture if mass is taken to be the rest mass of the matter fields. Thus let us focus on the 'if' part in the following. In [23] the authors show the Hoop conjecture holds in the spherically symmetric and time-symmetric situation for the mass being Brown-York mass (which is two times the asymptotic mass). This analysis has been extended in [19] to the non-spherically symmetric but still time-symmetric situation, with stronger geometric conditions. For completeness, see also [34] for interesting related ideas.

Usually, the Hoop Conjecture is treated as a conjecture independent of a specific matter model. This is mainly due to the fact that one defines the notion of 'horizon' via future null infinity in an asymptotically flat spacetime and therefore focuses on the *exterior* of the concentration region, where usually the Einstein-vacuum equation is supposed to hold (a notable exception here is the article [28] by Ponce de Leon giving a counterexample for Maxwell-fluid equations). In the following, we want to change the viewpoint slightly. There is a standard definition of 'black hole' for asymptotically flat spacetimes, generalized in [21] to arbitrary globally hyperbolic spacetimes. According to that latter definition, a point in  $(M, g)$  is called *black* iff there is no future timelike curve of infinite length starting at  $p$ , and the *black hole*  $\text{BH}(M, g)$  of  $(M, g)$  is the subset of all black points. Of course,  $\text{BH}(M, g)$  can be empty. In the one-ended asymptotically flat case, this notion coincides with the definition  $\text{BH}_M := M \setminus I^-(\mathcal{J}^+)$  where future null infinity  $\mathcal{J}^+$  is the set of ideal end points of null geodesics of infinite affine parameter, see Section 5. Moreover, this

notion of 'black hole' is exactly the one appearing in the conclusion of Hawking's singularity theorem.

As, by using the above definition of 'black hole', we are now equally interested in the behaviour in the *interior*, thus we have to take into account the field equations as well. The classical Oppenheimer-Snyder result in 1939 on the spherically symmetric Einstein-dust case can be considered the first matter-dependent affirmative result for the hoop conjecture in this sense. As e.g. solutions of Dirac-Einstein theory do not satisfy positive energy properties, which are an important element of the usual arguments, we focus on Maxwell-Einstein theory in the following. One reason for this choice is that Maxwell theory is considered to be a relatively fundamental theory (one building block of the Standard Model), which can be assumed to leave its domain of validity at a much larger energy scale than effective models like dust or other fluid equations. This is of some importance as due to the definition of blackness we must have the ambition to include the high-energetic interior of the black hole in our model. Moreover, Einstein theory (gravity) and Maxwell theory (electromagnetism) are *the only* long-range fundamental interactions that can be considered well-tested as classical field theories, and their canonical coupling, Einstein-Maxwell theory, has also been shown to be relevant as a classical field theory, e.g. in modelling the early, radiation-dominated, universe. Its Lagrangian density  $L : C^\infty(\tau^*M \otimes_{\text{sym}} \tau^*M) \times \Omega^1 M \rightarrow \Omega^n M$  is  $L(g, A) := \text{scal}^g \cdot \text{vol}^g + F \wedge *F$  for  $F := dA$  (in the trivial sector, i.e., where the bundle is trivial and connection is isotopic to the trivial one). The corresponding Euler-Lagrange equations are

$$d^*dA = 0, \quad \text{Ric}^g - \frac{1}{2}\text{scal} \cdot g = T(g, A)$$

with  $T(g, A)(e_i, e_j) = \frac{1}{4\pi}(\sum_a(F_{ai}F_j^a) - \frac{1}{4}\sum_{ab}(F^{ab}F_{ab})g_{ij})$ . The reasons above explain why it is considered an important issue to understand the space of solutions of Einstein-Maxwell theory, e.g. their timelike completeness.

The first toolbox for our purposes is the one of *conformal extensions*, this notion being a slight generalization of the one of conformal compactification. To understand necessity of generalization here, we first in Section 2 confirm folk knowledge by giving a result (Theorem 8) on nonexistence of sufficiently smooth conformal compactifications. Then we define the notion of *conformal extension* of order  $k$ : an open conformal embedding  $F$  into another globally hyperbolic manifold  $(N, h)$  with  $F$  and  $h$  of regularity  $C^k$  such that the closure  $\text{cl}(F(M))$  of its image is causally convex and future compact. This notion has been introduced in [15] where it proved useful to show that on conformally extendible spacetimes, Dirac-Higgs-Yang-Mills (DHYM) theories have a well-posed initial value problem for small initial values<sup>1</sup>. Let us define, for every null geodesic  $c$  in a Lorentzian manifold  $(N, h)$ , a Riemannian metric  $h_c$  along  $c$  by  $h_c = h + h(K, \cdot) \otimes h(K, \cdot)$  where  $K := \dot{c} + w$  and  $w$  is the unique null vector such that  $(\dot{c}, w)$  can be completed to a null frame. A *strong conformal extension* is a conformal extension with the property that the inverse  $\omega$  of the conformal factor, as a function on  $F(M)$ , can be extended to a  $C^k$  function on  $N$ , and such that along every past  $C^0$ -inextendible null geodesic along  $\partial^+ F(M)$ , we have that  $|(\text{Hess}(\sqrt{\omega}) - h)|_{\text{span}(\dot{c}, w)}|_{h_c} < \frac{1}{4}$ . We show in section 2 that such an object exists not only for slight perturbations of Minkowski space, but for the entire asymptotic class considered in Zipser's PhD thesis [37] where admissible initial values (furthermore called *Zipser-asymptotically flat*, for the details see Section 2) are defined via weighted Sobolev spaces. In the corresponding statement below, a spacetime is called to have *standard spatial infinity* if and only if for every future

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<sup>1</sup>modulo gauges, and each choice of a Cauchy surface corresponds to a smooth Cauchy section of the gauge orbits.

curve  $c : I \rightarrow M$  with  $J^-(c(I)) \cap S$  noncompact for a Cauchy hypersurface  $S$ , then all null future geodesic rays with image in  $J^-(c(I))$  that are  $C^0$ -inextendible in  $M$  have infinite affine length.

**Theorem 1.** *If  $I := (S, g_0, K_0, A_0, \dot{A}_0)$  are Zipser-asymptotically flat initial values for Einstein-Maxwell theory then the maximal Cauchy development  $(M, g, A)$  of  $I$  admits a strong conformal extension of order  $k$ . Moreover,  $(M, g)$  has standard spatial infinity, and  $J^+(S) \setminus J^+(C)$  has infinite timelike diameter for all compact  $C \subset S$ .*

This result will follow by an application of the Friedrich hyperboloidal criterion in the version of [1] to the stability result by Zipser. We will show how conformal extensions can serve as criteria for isometric maximality.

The second important toolbox is the one of singularity theorems. It turns out that we have to modify both classical singularity theorems slightly to reach our goal. Penrose's singularity theorem [27] is one of the cornerstones of mathematical relativity. It needs three conditions: Existence of a noncompact Cauchy surface in the spacetime, the *timelike convergence condition (TCC)*  $\text{ric}(v, v) \geq 0$  for all  $v \in TM$  causal (satisfied by all solutions of many Einstein-matter models like Einstein-Maxwell Theory), and the existence of a trapped compact spacelike codimension-2 submanifold  $N$ . Here, a submanifold  $N$  is called *trapped* iff its mean-curvature vector field is past. It is called *uniformly trapped* iff it is trapped and if there is  $c > 0$  s.t.  $g(H_N, H_N) < -c^2$ . Of course, any compact trapped submanifold is uniformly trapped. Under these three conditions, Penrose's theorem states that the spacetime contains an incomplete  $C^0$ -inextendible future lightlike geodesic starting at  $N$ . Still, to connect this to the usual idea of 'black hole', one could ask if there are also complete null geodesics from the trapped submanifold. Moreover, one is more interested in timelike than in null curves (as the former describe possible trajectories of massive objects, like observers). Now, answering the question whether a trapped submanifold implies the appearance of a black hole would imply an analogue of Penrose's theorem for *timelike* curves instead of null curves. There is an appropriate singularity theorem for timelike curves, namely Hawking's theorem [16]. Here we assume the existence of a uniformly trapped *Cauchy surface* (which is necessarily of codimension 1). Unfortunately, in the physically realistic spatially asymptotically flat case there is no uniformly trapped Cauchy surface, thus Hawking's theorem is not applicable to this important class of models. Therefore it would be desirable to have at our disposal a theorem for the asymptotically flat case whose hypothesis is of Penrose's type and whose conclusion is of Hawking's type. Such a synthesis we will indeed obtain for Maxwell-Einstein solutions in Theorem 4. First, in Section 3 we show, using Hawking's singularity theorem and a deformation argument:

**Theorem 2.** *Let  $(M, g)$  be globally hyperbolic spacetime satisfying the timelike convergence condition. If  $J^-(J^+(p))$  is spatially compact for a point  $p \in M$ , then  $p$  is black. The maximal future length is bounded on compact sets of black points.*

As the spatial compactness of  $J^-(J^+(p))$  is a purely conformal notion, whereas blackness of points depends on the full geometry, this theorem provides an interesting connection between the conformal structure of a spacetime and its full geometry, and the bridge between those two levels is unsurprisingly given by a condition on the Ricci tensor. Note also that for a precompact open set  $U$ , if  $J^-(J^+(\partial U))$  is spatially precompact, then also  $J^-(J^+(\bar{U}))$  is spatially precompact.

Furthermore, in section 4, we get a slight generalization of Penrose's singularity theorem, Theorem 16, replacing trapped surfaces by outer trapped surfaces.

Schoen and Yau, in their remarkable paper [32], consider the following setting: For an open set  $U$  of a Riemannian manifold  $S$ , denote by  $K_U$  the space of simple closed curves in  $U$  that are contractible in  $U$ , and for  $k \in K_U$  define

$$\text{Rad}(U, k) := \sup\{r > 0 \mid d(k, \partial U) > r, k \notin K_{B_r(k)}\}, \quad \text{Rad}(U) := \sup\{\text{Rad}(U, k) \mid k \in K_U\}.$$

It is easy to show that a Euclidean ball  $B_r(p)$  has  $\text{Rad}(B_r(p)) = r/2$ , and for a round sphere  $\mathbb{S}^2(r)$  of radius  $r$  we get  $\text{Rad}(\mathbb{S}^2(r) \times (-L, L)) = \min\{\frac{\pi r}{2}, L\}$ . For Maxwell-Einstein initial values  $I := (S, g_0, K_0, A_0, \dot{A}_0)$ , we define  $\mu \in C^\infty(S)$  and  $J \in \Omega^1(S)$  by

$$\mu := \frac{1}{2}(\text{scal} - \sum_{i,j} h^{ij} h_{ij} + (\sum_i h_i^i)^2), \quad J^i = \sum_j \nabla_j (h^{ij} - (\sum_k h_k^k) g^{ij}).$$

We say that  $I$  satisfies the *Schoen-Yau concentration condition* or is *Schoen-Yau concentrated* iff there is an open precompact  $U \subset S$  such that

$$\text{SY}(g, h, U) := \text{Rad}(U) \cdot \sqrt{\min\{\mu(q) - \|J(q)\| : q \in U\}} \geq \sqrt{\frac{3}{2}}\pi.$$

Schoen and Yau show in [32] that if  $I$  is Schoen-Yau concentrated then it contains a MOTS (and by an application of a result of Galloway [12] of an outer trapped<sup>2</sup> surface). Taking together Theorem 16 with Schoen-Yau's result, we obtain,  $MCD(I)$  being the maximal Cauchy development of the initial value  $I$ :

**Theorem 3.** *Let  $L$  be the Lagrangian of any field theory satisfying the null energy condition, let the Schoen-Yau concentration condition be satisfied in an open subset  $U$  of a noncompact Cauchy surface of a spacetime  $(M, g)$ , then there is a  $C^0$ -inextendible incomplete null geodesic in  $MCD(I)$  starting from  $U$ .*

Finally, in Section 5, the results established before allow us to obtain the announced synthesis between Penrose's and Hawking's theorem via a detailed analysis of a related semilinear symmetric-hyperbolic equation near the conformal boundary:

**Theorem 4.** *Let  $(M, g, A)$  be the maximal Cauchy development for Zipser-asymptotically flat Maxwell-Einstein initial values on  $S$  containing an outer trapped surface  $N$  that bounds a compact subset  $U$  of  $S$ . Then  $U$  is black, and thus,  $U \subset M \setminus J^-(\mathcal{I}^+)$ .*

This theorem, combined with Schoen-Yau's result, then implies:

**Theorem 5.** *Let  $\dim(S) = 3$ , let  $I := (S, g_0, K_0, A_0, \dot{A}_0)$  be Zipser-asymptotically flat Schoen-Yau concentrated initial values for Maxwell-Einstein theory. Then  $MCD(I)$  contains a nonempty black hole.*

Note that by the constraint equations we have  $\mu = T_A(\nu, \nu)$  and  $J = T_A(\nu, \cdot)$  for the Maxwell energy-momentum at the one-form  $A$ , and the dominant energy property of Maxwell theory implies  $\mu - \|J\| \geq 0$ , with equality holding only for those points  $p$  with  $F^A(p) = 0$ , and that  $T_A$  is homogeneous in  $F^A$ . Thus, given any Zipser-asymptotically flat initial value  $I := (S, g_0, K_0, A_0, \dot{A}_0)$  and any open precompact subset  $U \ni q$  of  $S$  with  $F^A(q) \neq 0$ , there is  $E_0 > 0$  such that the Schoen-Yau concentration condition in  $U$  is satisfied for the Zipser-asymptotically flat initial value  $(S, E^2 \cdot g_0, K_0, E \cdot A_0, \dot{A}_0)$  for all  $E \geq E_0$ .

In Section 6, we briefly discuss possible physical interpretations of the results.

<sup>2</sup>The term 'trapped surface' in Theorem 2 of [32] probably refers to 'outer trapped surface', as apparent horizons depend on the chosen surrounding Cauchy hypersurface and do, as far as I can see, not allow for concluding trappedness as defined above in general. Also, Schoen-Yau's result concludes existence of a MOTS, and Galloway's result, within its proof, states that either the MOTS can be deformed to an OTS (of positive Yamabe type) or else is the boundary of an infinite cylinder, which is impossible in our case due to precompactness.

## 2. CONFORMAL COMPACTIFICATIONS AND EXTENSIONS, AND HOW TO USE THEM TO DETECT ISOMETRICALLY INEXTENDIBLE SPACETIMES

Let  $k \in \mathbb{N} \setminus \{0\}$ . A subset  $C$  of a spacetime  $P$  is called *causally convex* iff any causal curve intersects  $C$  in the image of a (possibly empty) interval, i.e. no causal curve can leave and re-enter  $C$ , and  $C$  is called *spatially precompact* iff, for any closed acausal set  $B$ , the set  $\overline{C} \cap B$  is compact. Now, let  $(M, g)$  and  $(N, h)$  be globally hyperbolic manifolds. An open conformal embedding  $F \in C^k(M, N)$  is said to *extend  $g$  conformally* or to be a *conformal compactification of  $(M, g)$*  iff  $\text{cl}(F(M))$  is causally convex and spatially precompact. It follows automatically that  $F(M)$  is contained in the domain of dependence of a compact set: If  $S$  is any Cauchy set of  $(M, g)$ , causal convexity of  $F(M)$  implies that  $F(S)$  is an acausal  $C^k$  hypersurface, consequently, using [4], one can extend it to a  $C^k$  hypersurface  $T$  of  $(N, h)$ , and the closure of  $F(S)$  in  $T$  satisfies the requirement.

Our starting point is the following easy statement on possible conformal compactifications of asymptotically flat Riemannian manifolds. It is probably well-known to experts on the field, but I could not find a written statement in the literature.

**Theorem 6.** *Let  $(S, g_0, 0)$  converge to standard initial values to order 0, i.e.  $\|(g_0)_{ij} - \delta_{ij}\| < c/r$  for an appropriate constant  $c > 0$ , where  $r$  is the Euclidean radius. Then every open conformal embedding of  $(S, g_0)$  is a map into a sphere the complement of whose image is exactly one point.*

*Proof.* We show that, for every Riemannian manifold  $(N, h)$  and every conformal extension  $F : M \rightarrow N$  with  $h = u \cdot F_*g_0$  on  $F(M)$ , the topological boundary of  $F(S)$  consists of one point only. To that purpose, let  $x_1, x_2 \in \partial F(M)$  be given. We want to show that their  $h$ -geodesic distance vanishes. Consider sequences  $y_n^i \in M$  converging to  $x_i$ , for  $i \in \{1, 2\}$ . We choose accumulation points  $s_i$  of the projections  $p_n^i$  of  $y_n^i$  on the  $\mathbb{S}^{n-1}$  component (w.r.t. the diffeomorphism  $P : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^+ \times \mathbb{S}^{n-1}$  induced by  $r$ ). Then the curve  $k_i$  defined by  $k_i(t) := P^{-1}(s_i, t)$  has finite length, and an easy triangle inequality argument shows that  $k_i$  approaches  $x_i$ . Now let  $\epsilon > 0$  and choose points  $y_i$  in the image of  $k_i$  with  $d(x_1, y_1) < \epsilon/6$  and  $d(x_2, y_2) < \epsilon/6$ . What remains to be shown is that there is a curve of length  $< \epsilon/3$  from  $y_1$  to  $y_2$ . To that aim, choose an arc-length parametrized geodesic curve  $\gamma$  of length  $\leq 2\pi$  in  $\mathbb{S}^{n-1}$  between  $s_1$  and  $s_2$ . Now, the curves  $k_s$  (for simplicity all parametrized by  $g$ -arc length) are of finite  $h$ -length, thus for all  $r_0$  there is an  $r > r_0$  such that  $u(k_s(r)) < \frac{\epsilon}{12\pi}$ . Also for each  $s$  there is an  $r_1(s) > 0$  such that the  $h$ -length of  $c_s|_{[r_1, \infty)}$  is smaller than  $\frac{\epsilon}{6}$ . By compactness, there is a finite maximum  $m$  of  $r_1$  along the sphere. Now we want to construct a continuous and piecewise  $C^1$  curve  $\hat{\gamma}$  of length smaller than  $\epsilon/3$  projecting to  $\gamma$ . The strategy is to begin at a point  $P^{-1}(v, \gamma(0))$  for some  $v > m$  and then to use the open neighborhoods  $U_l$  in which  $u(k_s(r)) < \epsilon/r^2$  to progress along the sphere (i.e., with constant first component) up to the boundary of  $U_l$  and then continue in outward radial direction (i.e., with constant second component) until reaching the next neighborhood  $U_{l+1}$ . An easy connectedness argument ensures that using this procedure, one can cover the whole path  $\gamma$ . The lengths of the spherical parts of  $\hat{\gamma}$  add up to at most  $\epsilon/6$  as do the radial parts.  $\square$

**Theorem 7.** *Let  $(M, g)$  and  $(N, h)$  be given. A conformal extension of  $(M, g)$  into  $(N, h)$  is uniquely given by its values on a  $g$ -Cauchy set  $S$ .*

*Proof.* The core step of the proof is that any point in the future of  $S$  is given uniquely in terms of the intersection of its past with  $S$ . This can be proven along the lines of Theorem 3 in [21]: Given  $p, q \in M$ ,  $p \notin J^-(q)$ , then there is a  $C^0$ -inextendible past timelike past curve  $c$  from  $p$  not intersecting  $J^-(q)$ . As the details of the proof are a bit different from the one of the similar statement in [21], let us include a proof here: Let  $q^+ \in I^+(q)$  and consider, for a Cauchy time function  $t$  on  $M$ , the sets



$S_a := t^{-1}(a)$  and  $J_a := (I^-(p) \setminus J^-(q^+)) \cap S_a$ . First of all, the sets  $J_a$  are nonempty, for all  $a < t(p)$ : If  $I^-(p)$  and  $J^-(q)$  have empty intersection in  $I^+(S_a)$  then there is nothing to show. If they do intersect, then we choose  $y \in I^-(p) \cap \partial J^-(q^+) \cap I^+(S_a)$ . By compactness of  $J^-(q^+) \cap J^-(S_a)$  we can use elementary neighborhoods to show that there is  $x \in S_a \cap \partial J^-(y) \cap \partial J^-(q^+)$ . Thus  $x \in J^-(I^-(p)) = I^-(p)$  by the push-up lemma. Openness of  $I^+(p)$  and the fact that  $x \in \partial J^-(q^+)$  show that  $J_a$  is nonempty, containing an element  $z_a$ . Now choose past causal curves  $c_n$  from  $p$  to  $z_n$ . The limit curve lemma implies that there is a limit curve in the closure of  $M \setminus J^-(q^+)$ , which is  $M \setminus I^-(q^+)$ . As in turn  $J^-(q)$  is a spatially compact subset of  $I^-(q^+)$  by Theorem 3 in [21], the claim follows.  $\square$

**Theorem 8.** *Let  $(S, g_0, K_0, B_0)$  be asymptotically flat initial values for matter-Einstein equations obeying the dominant energy condition,  $S$  spin or of dimension  $\leq 7$  and with not identically vanishing matter fields  $B_0$ , then any open conformal embedding with precompact image is not  $C^2$  at infinity.*

*Proof.* First, theorem 6 implies that any compactification would be by one point  $i_0$  only. Next, the appropriate spacetime positive mass theorem [36], [26], [11] implies that the ADM mass does not vanish. Now we examine the conformal invariants  $W \in C^\infty(M)$  defined by  $w(x) := \sum_{abcd} \text{Weyl}^{abcd}(x) \text{Weyl}_{abcd}(x)$  and  $W \in C^\infty(\mathbb{R}^+, \mathbb{R})$ ,  $W(r) := \int_{\partial B(p,r)} w / \text{vol}_{n-1} \partial B(p,r)$ , and slightly adapt (from  $w$  to  $W$ ) the example of Ashtekar-Hansen [2] showing that, as in a standard Schwarzschild slice we have  $w \in \frac{M}{r^6} + O(r^{-7})$ , in asymptotically flat manifolds we still have  $W \in \frac{M}{r^6} + O(r^{-7})$  and for  $g' = a^2 g$  with  $a \in \frac{1}{r^2} + O(r^{-3})$  we get  $W' \in r^2 + O(r)$ , thus  $W'$  diverges towards  $i_0$ . Alternatively, reduce the general case to the Schwarzschild calculation in [2] by employing Lemma 2 of [33] (which in turn refers for some proofs to [31] and to [5]), generalized to appropriately defined *almost* conformally flat manifolds (referring to blown-up Riemannian normal neighborhoods of  $i_0$ ) via a  $3\epsilon$ -argument involving the tensors  $K_0, B_0$ .  $\square$

This implies that one has to generalize the notion of conformal compactification if one wants it to be at least  $C^2$ . Our choice is the one of *conformal extension*. A conformal extension is an open conformal embedding  $F : (M, g) \rightarrow (N, h)$  between two globally hyperbolic manifolds such that  $\text{cl}(F(M))$  is causally convex and *future compact*. Here a subset of a globally hyperbolic manifold is called *future compact* iff its intersection with every future set is compact, or equivalently, iff it is contained in the past of a compact set. It has been shown in [15] that conformal extensions can be very useful for PDE questions, e.g. it leads to well-posedness of the small-initial value problem in Dirac-Higgs-Yang-Mills theories. The usual way around the problem at spatial infinity shown above is to just renounce higher regularity of the metric of  $(N, h)$  at  $i_0$ . If in such a generalized conformal compactification  $E : M \rightarrow N$ , weaker differentiability at  $i_0$  is permitted then we can construct a conformal extension in the sense above by simply replacing  $N$  by  $I^+(E(M))$ , which is a future set in  $N$  and thus globally hyperbolic. In this sense the notion of conformal extension is more generally applicable, still offering the same analytical benefit. Theorem 1, which we prove now, shows the abundance of conformally extendible spacetimes. Here, Einstein-Maxwell initial values  $(S, g_0, K_0, A_0, \dot{A}_0)$  are called *Zipser-asymptotically flat (of mass  $m$ )* iff  $\text{tr}^{g_0}(K_0) = 0$  and

- $(g_0)_{ij} \in (1 + 2m/r)\partial_{ij} + o_4(r^{-3/2})$ ,
- $(k_0)_{ij} \in o_3(r^{-5/2})$ ,
- $(F(A_0, \dot{A}_0))_{ij} \in o_3(r^{-5/2})$ .

Zipser's theorem states that Zipser-asymptotically flat initial values satisfying a global smallness assumption (satisfied always after appropriate rescaling) have a

causally complete maximal Cauchy development that can be foliated by maximal Cauchy hypersurfaces. The assumption of maximality ( $\text{tr}^{g_0}(K_0) = 0$ ) is merely a technical assumption that could be renounced by invoking the existence of a foliation by maximal hypersurfaces and an application of an inverse function theorem.

*Proof of Theorem 1.* We want to apply Theorem 4.1 and Theorem 6.1 of [1] to the hyperboloidae of the maximal Cauchy development in coordinates given by a maximal temporal function. Note that we cannot apply 4.1 directly to the initial Cauchy hypersurface  $S$ , as the positive mass theorem forbids a  $C^2$  extension (cf. Theorem 8). What we can do, however, is to consider the sequence of 'hyperboloidal' subsets  $S_k := \{x \in M \mid -(t(x) + k)^2 + r^2 = k^2\}$  w.r.t. the coordinates  $r, t$  defined by Zipser. Using the boundedness of the weighted quantities  $W, K, L, O$  in Zipser's article, it is lengthy but straightforward to see that suitable conformal multiples of the metrics are indeed  $C^l$  extendible, allowing for a conformal extension of  $D^+(S_k)$ . Uniqueness of the associated symmetric-hyperbolic system in the harmonic-coordinates constant-scalar-curvature (hccsc) gauge implies that all these conformal extensions coincide (in the chosen constant scalar curvature and harmonic gauge) on the intersection of their domains of definition. To show the required behaviour at  $i_0$ , we use that there is an  $\epsilon > 0$  s.t. if initial value  $a$  has  $\|a\|_{C_w^k} < \epsilon$  for a weight  $w$  then the maximal solution has the required behaviour. Choose a higher weight  $u$  and an associated weighted space  $C_u^k$ , then every initial value  $b$  with  $\|b\|_u^k < \infty$  satisfies that there is a compact subset  $K \subset S$  such that for  $b_{\text{ext}} := b|_{S \setminus K}$  we have  $\|b_{\text{ext}}\|_{C_w^k} < \infty$ . Complement  $b$  on  $K$  to an initial value  $B$  s.t.  $\|B\|_{C_u^k} < \epsilon$ . Then the maximal Cauchy development of  $B$  has the required end behaviour, and by local uniqueness of Einstein-Maxwell equations, its end is isometric to  $D(S \setminus K)$  as a subset of the maximal Cauchy development of  $b$ .

Causal convexity of  $\overline{F(M)} = \overline{D(F(S))}$  is automatical, for strongness consider the symmetric hyperbolic equation satisfied by  $\omega = \Omega^{-1}$  again in hccsc gauge and use the conventional stability arguments. Note that in the Penrose compactification of Minkowski space in the Einstein cylinder ( $\mathbb{E} = \mathbb{R} \times \mathbb{S}^3, -dT^2 + g_{\text{round}}$ ) with the image  $\{x \in \mathbb{E} \mid 0 \leq \alpha(x) < \pi, \alpha(x) - \pi < T(x) < \pi - \alpha(x)\}$ , we have  $\omega = \beta^2$  for  $\beta := \cos \alpha + \cos T = 2 \cos \frac{\alpha+T}{2} \cos \frac{\alpha-T}{2}$ . The Hessian of  $\beta$  is a constant multiple of the metric on the orthogonal complement of the spherical orbits and thus  $\text{Hess} \omega = 2\beta \text{Hess} \beta + d\beta \otimes d\beta$  is positive definite in all null directions not tangential to the boundary in a neighborhood around  $J^+(i_0) = (T + \alpha)(\pm\pi)$ , therefore  $C^2$ -stability in those coordinates implies the claim.  $\square$

A famous obstruction for conformal extensions are the peeling estimates due to Sachs [29] refined by Newman and Penrose [25], using null congruences. Other obstructions have been found in [7] and [22].

To show how to use conformal extensions to find sufficient criteria for isometric inextendibility, which is an important notion in the context of maximal Cauchy developments, let us consider Exercise 7.5.8. in the book by Sachs-Wu [30]:

*Show that Kruskal spacetime is (isometrically) inextendible, i.e. there is no open isometric embedding of Kruskal spacetime in a larger Lorentzian manifold!*

Whereas Sachs-Wu do not give a solution, in O'Neill's book [24], Corollary 13.37 also states that Kruskal spacetime is inextendible. For the proof, the book refers to its Remark 5.45 stating that

*If  $(M, g)$  is a connected semi-Riemannian manifold such that for every inextendible geodesic  $c : [0, b) \rightarrow M$  of given causal character, there is a nonconvergent curvature*

invariant  $I_c$  (e.g.,  $I_c := \text{Ric}(\dot{c}, \dot{c})$  or  $I_c := g(X, \dot{c})$  for a parallel vector field  $X$ ), then  $(M, g)$  is inextendible.

Two comments are in order: First, to apply this corollary, we need to assume that  $b < \infty$ , that is, we can restrict to geodesics of finite affine parameter, as for the geodesics escaping to null future infinity, all curvature invariants do converge. Second, we interpret 'of given character' in the sense that we can restrict the entire analysis to a prescribed causal character, spacelike, timelike, or null. Once accepting the corollary, in our situation, we could then take  $I_c := \text{scal} \circ c$  for all  $c$ , as  $\text{scal}_p = 48m^2/r^6(p)$  in Kruskal spacetime. Then we only have to show that every timelike geodesic  $c$  either satisfies  $l(c) = \infty$  or  $r \circ c \rightarrow 0$ , which is shown in O'Neill's book in Proposition 13.36 via a detailed analysis. How to prove the remaining Remark 5.45? Again no proof is given in [24], so let us first define, for  $A \subset M$ ,

$$\partial^\pm A := \{x \in \partial A \mid J^\pm(x) = \emptyset\}$$

and note that for a causally convex subset  $A$ , we have

$$\partial^\pm A = \{x \in \partial A \mid A \cap J^\mp(x) \neq \emptyset\}$$

Then one can state the following lemma, implying that the behaviour at  $i_0 \in \partial F(M) \setminus (\partial^+ F(M) \cup \partial^- F(M))$  is quite exceptional:

**Lemma 9.** *Let  $F : (M, g) \rightarrow (N, h)$  an open conformal embedding of connected spacetimes, then  $\partial^+ F(M)$  and  $\partial^- F(M)$  are open subsets of  $\partial F(M)$  and  $\partial F(M) = \partial^- F(M) \cup \text{cl}(\partial^+ F(M)) = \partial^+ F(M) \cup \text{cl}(\partial^- F(M))$ .*

*Proof.* Let  $x \in \partial F(M) \setminus (\partial^+ F(M) \cup \partial^- F(M))$  be given. Then every neighborhood  $U$  of  $x$  contains a neighborhood  $V = I^+(x^-) \cap I^-(x^+)$  of  $x$  for two points  $x^\pm \in I^\pm(x)$ . As  $x \in \partial F(M)$ , there is  $u \in V \cap F(M)$ , and as  $x \notin \partial^\pm F(M)$ , we have that  $u \in J^\pm(x)$ . Let  $c$  be a future causal curve from  $u$  to  $x^+$  and look for the first point  $q$  on it not contained in  $F(M)$  any more. Obviously,  $q$  is in  $\partial^+ F(M) \cap U$ . As  $U$  was an arbitrary open neighborhood of  $x$ , this shows the claim.  $\square$

Coming back to Kruskal manifold, let us ignore for a moment that, looking at the conformal diagram of Kruskal spacetime (which is really an orbit diagram of the isometric  $SO(3)$  action), we know exactly all TIPs and TIFs, allowing already for the conclusion that in fact Kruskal spacetime is isometrically inextendible. Let us instead assume that we do not are necessarily in the situation of spherical symmetry, but have a conformal extension of the spacetime. Keeping in mind that a conformal extension organizes the TIPs of a spacetime in a very visual way — they simply correspond to points on the future boundary — it should be of no great surprise that it can serve as a tool to show isometric *inextendibility*.

**Theorem 10.** *Let  $(M, g)$  be a globally hyperbolic spacetime and let  $F : (M, g) \rightarrow (K, k)$  be a conformal extension. If for each element  $a$  of  $\partial_K^+ M$  there is a geodesic curve  $c_a$  ending at  $a$ , such that either the affine length function or some curvature invariant  $k_a$  along  $c_a$  that does not converge, then  $(M, g)$  is inextendible.*

*Proof.* Assume there is an isometric extension  $E : (M, g) \rightarrow (N, h)$ . Then let  $p \in \partial^+ E(M)$  and consider the past null geodesics emanating from  $p$ . Their restrictions to  $M$  are future inextendible geodesic curves, with the same past, thus the restriction to  $M$  of one of them equals a curve  $c_a$  along which a curvature invariant or the length function diverges. On the other hand,  $E \circ c_a$  is a continuous (and thus smooth) extension of a geodesic curve in the isometric extension, contradiction.  $\square$



Now our question whether Kruskal spacetime is isometrically extendible has turned into the question whether it admits a *conformal* extension. Note that this does not follow directly from Theorem 1 as the Cauchy surfaces of Kruskal spacetime are not asymptotically flat in the sense of Zipser, as they are spatially two-ended, a notion we define as follows: Let a *spatial end* of a spacetime  $(M, g)$  be a connected component of the complement of a spatially compact subset. Any two spatial ends are called *equivalent* iff they have nonempty intersection. A spacetime is called *spatially  $k$ -ended* iff the space of such equivalence classes contains exactly  $k$  elements. In particular,  $(M, g)$  is one-ended iff any two spatial ends of  $(M, g)$  have nonempty intersection. In spite of not arising from Zipser-asymptotically flat initial values, Kruskal spacetime does admit a conformal future extension, as its metric is given as  $g = \frac{32M^3}{r} e^{-\frac{r}{2M}} g_{1,1} \oplus r^2 g_{\mathbb{S}^2}$ , on  $U \times \mathbb{S}^2$ , with  $U := \{(t, x) \in \mathbb{R}^2 | x^2 - t^2 > -1\}$  and  $r := 2M(1 + W(e^{-1}(x^2 - t^2)))$  for  $W$  being Lambert's  $W$  function. Obviously this is conformally extendible if and only if the two-dimensional metric  $h := r^{-2}g = \frac{32M^3}{r^3} e^{-\frac{r}{2M}} g_{1,1}$  is extendible. This latter metric is conformally equivalent to the 2-dimensional Minkowski metric, which is (by the above uniquely) conformally extendible by the appropriate restriction of the usual Penrose compactification into the Einstein cylinder. The extension is strong, as the conformal factor is, up to constants, just  $r \cdot e^{r/2M} \cdot A$  where  $A$  is the conformal factor of the Penrose compactification. Therefore Kruskal spacetime is strongly conformally extendible.

### 3. PROJECTION DIAGRAMS, VISUAL COMPACTNESS AND A USEFUL VARIANT OF HAWKING'S SINGULARITY THEOREM

In this section, we want to use the dominant-energy condition to connect the conformal structure of the spacetime to its metric structure. We need a deformation lemma for maximal hypersurfaces appearing already in a proof of Burnett [6]:

**Lemma 11.** *Let  $\epsilon > 0$ , let  $S$  be a maximal partial Cauchy hypersurface of a globally hyperbolic manifold  $(M, g)$  and let  $C$  be a spatially compact subset of  $M$ . Then there is a hypersurface  $S'$  with the properties:*

- $S' \subset J^+(S)$  and  $d(S, S') < \epsilon$ ,
- $\partial S' = \partial S$ ,
- $H_{S'}|_{C \cap S'} < 0$  with respect to the future normal vector.

*Proof.* We consider a normal evolution  $F$  of the inclusion  $j$  of  $S$ , that is, a map  $F : [0, 1] \times S \rightarrow M$  with  $f|_{\{0\} \times S} = j$  and  $\partial_t f(t, s)$  is pointwise a multiple of the normal vector field  $\nu_{f_t(S)}$ , i.e., there is  $h \in C^\infty([0, 1] \times S)$  s.t.  $\nu = h \cdot F_* \partial_t$ . Then

$$\frac{\partial H}{\partial t} = -h(\langle II, II \rangle + \text{ric}(n, n)) + \Delta h.$$

The timelike convergence condition implies that for  $h > 0$  the right-hand side is nonnegative for all  $h$  with  $\Delta h \leq 0$ . Now we choose an open precompact subset  $K \subset S$  with smooth boundary such that  $C \cap S \subset K$ . Choose  $u(s) = h(t=0, s)$  such that  $\Delta u = -f \in C^\infty(S)$  with  $f > 0$  on  $K$  and  $u = 0$  on  $\partial K$ . Hopf's maximum principle implies that  $u > 0$  on  $K$ , thus  $f \geq c > 0$  on  $C \cap S$ . Thus  $\frac{\partial H}{\partial t} \leq -c$  on  $C \cap S$ , and, consequently, for small  $t$ , we have  $H(S_t \cap C) < 0$ .  $\square$

A point  $p$  in a spacetime  $(M, g)$  is called (*future*) *visually compact* iff for every future curve  $c : I \rightarrow M$  from  $p$ ,  $J^-(c(I))$  is spatially compact. Note that this is a conformally invariant notion (while blackness is not). However, Theorem 2 shows that the TCC connects the two notions.

*Proof of Theorem 2.* Let  $\epsilon > 0$  be given and let  $c$  be a causal curve from  $p$ . Choose a Cauchy hypersurface  $S_0$  containing  $p$ , then we can find an open precompact subset

$U$  of  $S_0$  with smooth boundary whose interior contains  $J^-(c(I)) \cap S_0$ . Following a theorem of Bartnik [3] we can find a maximal spacelike hypersurface  $S$  with  $\partial S = \partial U$ . Theorem 11 implies that this hypersurface can be smoothly deformed to a hypersurface  $S'$  such that  $\partial S' = \partial S$ ,  $d(p, S') < \epsilon$  and  $\partial S' \cap J^-(c(I))$  has past mean curvature vector. As  $J^-(c)$  is spatially compact,  $H \leq -b < 0$  on  $J^-(c(I)) \cap S'$ . But then Hawking's singularity theorem implies that any curve from  $S' \cap J^-(c(I))$  has length  $< b^{-1}$ , in particular  $l(c) < b^{-1} + \epsilon$ .  $\square$

Which spacetimes contain spatially precompact TIPs? Schwarzschild and Kruskal spacetime do — their black points are exactly those from which only future curves  $c : I \rightarrow M$  emanate with  $I^-(c(I))$  spatially precompact, as can be seen directly from their orbit diagram. A tool for deciding whether a spacetime contains spatially precompact TIPs are *projection diagrams*, a notion invented by Chruściel, Ólz and Szybka [8] in a beautiful approach to formalize the idea of what a Penrose diagram should be. Let  $(M, g)$  be a spacetime, then a *projection diagram*<sup>3</sup> of  $(M, g)$  is a submersive map  $\pi \in C^\infty(M, \mathbb{R}^{1,1})$  mapping timelike curves to timelike curves and such that for every timelike curve  $c$  in  $\pi(M) \subset \mathbb{R}^{1,1}$  there is a timelike curve  $C$  in  $M$  with  $c = \pi \circ C$ . The article [8] then gives examples of projection diagrams: First, the usual triangle diagram is a proper projection diagram of  $\mathbb{R}^{1,3} \setminus \{r = 0\}$ , whereas the usual two-dimensional diamond is a projection diagram for the entire spacetime  $\mathbb{R}^{1,3}$  but then the projection is not proper, as we mod out the translational symmetry of  $\mathbb{R}^2$  instead of the rotational one, where the fiber was  $\mathbb{S}^2$ . Here we are less interested in properness of the projection (compactness of the fibers) than in the *completeness of the fibers*: If all fibers are complete, we call the projection diagram *complete* (of course, properness implies completeness). This is the case for the diamond projection above but not for the triangle projection. We call a projection diagram *embedded* if there is a conformal embedding of  $\pi(M)$  with the metric induced from  $\mathbb{R}^{1,1}$  into  $(M, g)$  which is a right inverse of  $\pi$ . Let us call a spacetime SCOT if it contains spatially precompact TIPs. Then if  $(M, g)$  is SCOT and  $\pi$  is a projection diagram for  $(M, g)$ , then  $\pi(M)$  is also SCOT. If  $\pi$  is complete in the sense above, then SCOTness of  $(M, g)$  is equivalent to SCOTness of  $\pi(M)$ . In [8] it is shown that, for Schwarzschild spacetimes  $m > 0, a = 0$ , the usual half-hexagonal Penrose diagram is indeed a proper embedded projection diagram, and for  $a > 0$  it is shown:

**Theorem 12** (see [8]). *For every element of the entire slow Kerr-Newman family<sup>4</sup> with  $a > 0$  there is a proper embedded projection diagram onto a causal diamond in  $\mathbb{R}^{1,1}$ . One of the two null geodesic segments whose union is the future boundary of the diagram represents the inner, Cauchy horizon consisting of endpoints of curves of finite length. The part  $n_1$  between  $i_0$  and  $i^+$  of the other null geodesic segment  $n$  is future null infinity, whereas the part  $n_2$  of  $n$  to the future of  $n_1$  consists of ideal endpoints of curves of finite length.*  $\square$

Looking at the diagrams we get as a trivial corollary:

**Theorem 13.** *In the Kerr-Newman family, the members of the Schwarzschild family with  $m > 0$  are the only ones that contain spatially precompact TIPs.*  $\square$

Recall that in the classical Kerr-Newman manifolds the topology of the Cauchy surfaces is  $\mathbb{R} \times \mathbb{S}^2$ , and the OTSs do not bound a compact set. Furthermore note that i.g. diagrams do not correspond to conformal extensions, we will see examples.

<sup>3</sup>We specialize this notion to  $U = M$  in the terminology of [8] as we are primarily interested in globally hyperbolic manifolds and in black hole spacetimes which typically have the problematic symmetry axis  $r = 0$  at the boundary of the diagram.

<sup>4</sup>Here we refer to the case  $a \leq m$  where  $m$  is mass and  $a$  is angular momentum and to the maximal Cauchy developments of the initial values with one incomplete inner end and one outer asymptotically flat end. If we chose a slice with two asymptotically flat ends instead, every maximal Cauchy developments would consist of four blocks instead of two.

#### 4. A USEFUL VARIANT OF PENROSE'S SINGULARITY THEOREM

In this section we revise the usual prerequisites of the Penrose singularity theorem and derive a slightly different version. We first recall some classical facts about adapted Jacobi fields for the non-expert reader. Let  $N_1, N_2$  be spacelike submanifolds of  $M$  and let  $c : [0, E] \rightarrow M$  be a timelike geodesic from  $p_1 \in N_1$  to  $q \in N_2$  which is orthogonal to the submanifolds at the endpoints, i.e.  $\dot{c}(0) \perp_g T_{p_1}N_1$  and  $\dot{c}(E) \perp_g T_qN_2$ . A Jacobi field  $J$  along  $c$  is called  $(N_1, N_2)$ -*adapted* iff  $J(0) \in T_{p_1}N_1$ ,  $J(E) \in T_qN_2$ ,  $\text{proj}_g^{\text{tan}}(\nabla_t J(0)) = \hat{S}_p^{N_1}(J(0), \dot{c}(0))$  and  $\text{proj}_g^{\text{tan}}(\nabla_t J(E)) = \hat{S}_q^{N_2}(J(E), \dot{c}(E))$  where the  $\hat{S}$  denote the second fundamental forms of the respective submanifolds seen as bilinear maps  $TN_i \times T_g^\perp N_i \rightarrow TN_i$ . More explicitly, for all  $Y \in TN_1$  we have  $\langle \nabla_t J(0), Y \rangle = \langle \hat{S}^{N_1}(J(0), Y), \dot{c}(0) \rangle$ . The adaptedness is closely related to a similar property of geodesic variations, namely  $(N_1, N_2)$ -*properness*. A geodesic variation  $V$  of  $c$  is called  $(N_1, N_2)$ -*proper* iff  $V(s, 0) \in N_1$ ,  $V(s, E) \in N_2$ ,  $\partial_t V(s, 0) \perp TN_1$ ,  $\partial_t V(s, E) \perp TN_2$  for all  $s \in (-1, 1)$ . Analogously, in the obvious way, we introduce the notions 'initially adapted' and 'terminally adapted'. As an easy consequence of torsion-freeness of the Levi-Civita connection (pulled back to the domain of a variation) we get a well-known relation between the two notions: The variational vector field of an  $(N_1, N_2)$ -proper geodesic variation is an  $(N_1, N_2)$ -adapted Jacobi field. Conversely, if a Jacobi field  $J$  is  $(N_1, N_2)$ -adapted, then there is an  $(N_1, N_2)$ -proper geodesic variation with variational vector field  $J$ . If  $t \in (0, E)$  is such that the dimension  $A(N_1, c, t)$  of the vector space of  $(N_1, \{c(t)\})$ -adapted Jacobi fields is nonzero,  $t$  is called  $N_1$ -*focal point to 0 along c (of order  $A(N_1, c, t)$ )*. It is well-known that after the first  $N_1$ -focal point, a timelike geodesic cannot be maximal between the endpoint and the initial submanifold.

We first revise a well-known intermediate theorem holding without causality assumptions on  $(M, g)$ . To formulate it, we first need to quantify affine length of a null geodesic by an initial gauge, which is defined via scalar product with some timelike vector field  $V$  (which exists due to time orientation). Explicitly, for a spacelike codimension-two submanifold  $A$ , we set  $B_A := \{v \in \tau M^{-1}(N) | v \perp TN, g(v, v) = 0, g(v, V) = -1\}$ . Note that this is a subbundle of  $\tau M^{-1}(N)$  with zero-dimensional fiber. The fiber consists of two points, and it is easy to see that if  $N$  is a connected oriented submanifold then  $B_A$  is disconnected. For example, if  $A = \partial G$  for a spacelike hypersurface  $G$  then we have an inner and an outer null normal, so  $B_A$  is the disjoint union of  $B_A^+$  (outer null normals, generated by  $k_+ := \nu + n$ , where  $\nu$  is the timelike normal of  $G$  and  $n$  is the outer normal of  $A$  in  $G$ ) and  $B_A^-$  (inner null normals, generated by  $k_- := \nu - n$ ).  $A$  is called *outer trapped resp inner trapped* iff  $g(H_{A \subset M}, k_+) < 0$  resp.  $g(H_{A \subset M}, k_-) < 0$ , and as  $\nu = k_- + k_+$ , obviously  $A$  is trapped iff it is inner and outer trapped. Let us first, for greater self-containedness, revise the following complete analogue of a classical result:

**Theorem 14.** *Let  $(M, g)$  be an  $n$ -dimensional Lorentzian manifold and  $A \subset M$  a spacelike 2-codimensional submanifold of mean curvature  $H$ . Let  $C : [0, D) \rightarrow M$  be a null geodesic starting at  $p \in A$  and  $\dot{C}(0) \in B_A^+$ . If there is  $b > 0$  with  $g(H(p), \dot{C}(0)) \geq 1/b$  and  $\text{ric}(\dot{C}(t), \dot{C}(t)) \geq 0$  for all  $t \in [0, b]$ , then  $C$  is not  $A$ -maximal after  $b$ . Consequently,  $J^+(A) \setminus I^+(A) \subset \exp([0, b] \cdot B_A) \cap \text{Def}(\exp)$ .*

*Proof.* We first assume that  $C$  is  $A$ -maximal beyond  $b$  and restrict  $c := C|_{[0, b]}$ . It is an easy classical result (using the symmetry of the Riemann curvature tensor) that for any two nontangential Jacobi fields along a geodesic  $k$ ,  $g(\dot{J}_i, J_j) - g(J_i, \dot{J}_j)$  is constant along  $k$ . This implies the following well-known fact: Let  $J_1, \dots, J_l$  be nontangential Jacobi fields along a geodesic  $k : [0, E] \rightarrow M$ . Then if there are functions  $f_i$  on  $[0, E]$  such that  $X = \sum_{i=1}^l f_i J_i$ , we have

$$g(\dot{X}, \dot{X}) - g(R(X, \dot{k})\dot{k}, X) = g(L, L) + \frac{d}{dt}g(X, B)$$

where  $L := \sum_{i=1}^l \dot{f}_i J_i$  and  $B := \sum_{i=1}^l f_i \dot{J}_i$ .

Now we apply this fact to  $k = c$  and to  $A$ -adapted Jacobi fields  $J_i$  along  $c$  with  $J_i(0) = e_i$  for an orthonormal basis  $e_1, \dots, e_{n-2}$  of  $T_p A$ . The lemma implies that for every smooth vector field  $X$  along  $c$  with  $X(E) = 0$  and  $X(t) \in \text{span}(J_1(t), \dots, J_l(t))$  for all  $t \in [0, E]$ , we have

$$\int_0^b (g(\nabla_t X, \nabla_t X) - g(R(\dot{k}, X)X, \dot{k}))dt - g(X(0), \sum f_i \dot{J}_i) > 0$$

where the  $f_i$  are defined by  $X = \sum_{k=1}^l f_k J_k$ , because  $g(L, L) \geq 0$ . Now we apply this to the non-tangential vector fields  $X_i$  along  $c$  defined as  $X_i(t) := (1 - t/b)P_c^t(J_i(0))$ , obtaining  $\nabla X_i(t) = -\frac{1}{b}P_c^t(J_i(0))$ , and  $X_i(t) = \sum_{j=1}^{n-1} f_i^j(s)J_j(s)$ , with  $f_i^j \in C^\infty([0, b])$  and  $f_i^j(0) = \delta_i^j$ , and  $g(X(0), \sum f_i \dot{J}_i) = g(\dot{c}(0), S_N(X_i(0), X_i(0)))$  due to  $S$ -properness. So we get

$$0 < \int_0^b (g(\nabla_t X_i, \nabla_t X_i) - g(R(\dot{c}, X_i)X_i, \dot{c}))dt - g(\dot{c}(0), S_N(X_i(0), X_i(0))).$$

We choose  $a = 1$  and sum over  $i$ . Then we use the general fact that in an  $n$ -dimensional Lorentzian manifold  $(M, g)$ , for a null vector  $v \in T_p M$  and  $e_1, \dots, e_{n-2}$  spacelike unit vectors that are orthogonal to each other and to  $v$ , we get

$$\sum_{i=1}^{n-2} g(R(v, e_i)e_i, v) = \text{ric}(v, v). \text{ And we calculate for } e_i := P_c^t(J_i(0)):$$

$$\begin{aligned} 0 &< \frac{n-2}{b^2}b - \int_0^b (1 - \frac{t}{D_0})^2 \cdot \sum_{i=1}^{n-2} g(R(\dot{c}, e_i)e_i, \dot{c})dt - (n-2)g(\dot{c}(0), H(p)) \\ &= \frac{n-2}{b^2}b - (n-2)g(\dot{c}(0), H(p)) - \int_0^b (1 - \frac{t}{b})^2 \cdot \text{ric}(\dot{c}(t), \dot{c}(t))dt \\ &< \frac{n-2}{b} - (n-2)g(\dot{c}(0), H(p)) \end{aligned}$$

(where the last step uses the condition on the Ricci tensor), contradiction.  $\square$

It is worthwhile to compare this intermediate theorem to Proposition 1 of the beautiful recent paper [13], which has been an approach to unify singularity theorems using trapped surfaces of different codimensions. The crucial condition of that proposition is  $\text{tr}(g(R(\dot{c}, \cdot)\dot{c}, \cdot)|_{P \times P}) > 0$ , where  $P$  is the parallel transported  $TN$  along  $c$ . This condition does not follow from the timelike convergence condition, although the authors of [13] remark correctly (in Remark (ii) after that proposition) that in the case of codimension 2 the condition does follow from the null convergence condition *if the curve is null* (an assumption made implicitly by the authors of [13] by their notion  $n := \dot{c}(0)$ ).

**Theorem 15.** *Let  $(M, g)$  be a connected globally hyperbolic Lorentzian manifold and  $A \subset M$  a compact achronal spacelike outer trapped codimension-two submanifold of  $M$ . If  $\text{ric}_M(X, X)$  for all null vectors in  $J^+(A)$ , then either  $\partial^+(D(\text{ext}(A)))$  is compact, or there is a noncomplete null geodesic starting on  $A$ .*

*Proof.* Compactness of  $B_A^+$  implies that  $g(H_M, \cdot)$  attains a positive minimum  $1/b$  on  $B_A^+$ . As geodesics stop being  $A$ -maximal after the first  $A$ -focal point, either there is some incomplete null geodesic starting on  $A$  (so that  $\exp$  is not defined for a sufficiently long time) or each point  $q \in \partial^+(D(\text{ext}(A)))$  is of the form  $\exp_M(a \cdot v)$  for  $v \in B_A^+$  and  $a \leq b$ , thus  $\partial^+(D(\text{ext}(A)))$  is compact.  $\square$

**Theorem 16.** *If  $(M, g)$  satisfies the null energy condition and contains a noncompact Cauchy surface, which contains a compact subset  $U$  whose boundary is an outer trapped surface, then there is a  $C^0$ -inextendible incomplete null geodesic from  $\partial U$ .*

*Proof.* First we use the result by Bernal and Sánchez [4] that every compact spacelike acausal submanifold-with-boundary (as  $U$  is one) is a subset in a smooth Cauchy hypersurface to find a Cauchy hypersurface  $S$  of  $M$  containing  $U$ . For  $\text{ext}(A) := S \setminus U$ ,  $F := \partial^+(D^+(\text{ext}(A)))$  is nonempty. Furthermore,  $F$  is an achronal topological hypersurface. Finally, following Theorem 15,  $F$  is compact. We define a complete timelike future vector field  $V$  by the definition of future orientation and by rescaling to length one for a complete auxiliary Riemannian metric. Consider the map  $Q : F \rightarrow S$  given by the flow of  $V$ . It is well-defined as  $S$  is a Cauchy surface and thus achronal. Moreover, it is continuous as  $V$  is smooth. The image of  $Q$  lies in  $\text{ext}(A)$  by connectedness. We want to show surjectivity of  $Q$  onto  $\text{ext}(A)$ . First we observe that  $Q$  is injective as  $S$  is achronal. As  $F$  and  $\text{ext}(A)$  are of the same dimension, the famous Brouwer's Theorem of invariance of domain for manifolds tells us that  $Q(F)$  is open. All in all,  $Q(F)$  is compact, thus closed, and open, thus all of  $\text{ext}(A)$ . But this is in contradiction to  $\text{ext}(A)$  noncompact.  $\square$

It is worthwhile to compare Theorem 16 to Theorem 3.2 in the interesting article [10] by Eichmair, Galloway and Pollack, where the trapped surface in Penrose's original theorem is replaced by the assumption of existence of a MOTS, at the expense of additionally assuming the so-called 'generic condition' (whose genericity, from the initial data point of view is however unclear).

## 5. COMPARING DIFFERENT NOTIONS OF ASYMPTOTIC FLATNESS AND A BLACKNESS THEOREM

Let us compare different notions of asymptotic flatness and resulting notions of future null infinity. It is well-known that asymptotic simplicity is too strong a requirement for many purposes. On the other hand, weak asymptotic simplicity and emptiness is too weak a condition for, e.g., obtaining global existence of solutions to semilinear partial differential equations along the lines of [15]. We have argued that often it can be replaced by the assumption of the existence of a conformal extension, giving up the requirement that the conformal factor equals zero on the entire boundary, to be able to treat Cauchy horizons and future null infinity on the same footing. Thus let us compare those different notions now. We call a geodesic *infinite* iff its affine parameter is bounded. Otherwise we call it *finite*. We define  $LM$  to be the total space of the (nonlinear) subbundle of the null vectors in  $TM$ . The following two definitions are from the classical papers of Penrose, for a detailed introduction see the book of Hawking-Ellis [17]. The presentation of the following definition differs from that book, but is easily seen to be completely equivalent.

**Definition 17.** *An orientable spacetime  $(M, g)$  is called asymptotically simple iff it has a simple extension or Penrose extension, which in turn we define as a triple  $(\psi, \hat{M}, \hat{g})$  such that*

- (i)  *$(\hat{M}, \hat{g})$  is a strongly causal spacetime and  $\psi : M \rightarrow \hat{M}$  is a conformal embedding.*
- (ii) *No maximal future- or past-directed lightlike solution  $c : [0, T) \rightarrow \hat{M}$  of the geodesic equation satisfies  $c([0, T)) \subset \psi(M)$ ; we define  $\mathcal{J} := \mathcal{J}(\psi, \hat{g}) := \exp(D \cap L\psi(M)) \cap \partial\psi(M)$ .*
- (iii) *Let  $\Omega$  be the conformal factor for  $\psi$ , i.e.,  $\psi^*\hat{g} = \Omega^2 \cdot g$ . Then the function  $\omega := \Omega^{-1} \circ \psi^{-1}$  on  $\psi(M)$  extends to a function  $\hat{\omega} \in C^3(\hat{M})$  such that  $\hat{\omega}|_{\mathcal{J}} = 0$  and  $d\hat{\omega}|_{\mathcal{J}} \neq 0$ .*

Note that  $\mathcal{J}$  is never empty. The definition of asymptotic simplicity excludes interesting spacetimes — like Schwarzschild spacetime or even future causally complete spacetimes with a spacelike future infinity — which was the reason for Penrose's second definition:

**Definition 18.** A spacetime  $(M, g)$  is called weakly asymptotically simple iff there is a simple extension  $\psi : M' \rightarrow \hat{M}'$  and an open neighborhood  $\hat{U}'$  of  $\mathcal{J}(\psi, \hat{g}')$  such that there is an open subset  $U$  of  $M$  isometrical to  $U' := \psi^{-1}(\hat{U}')$ .

**Remark 19.** By intersecting  $U$  with the open set  $I^-(\mathcal{J}, \hat{M}')$  we can and will always assume that  $U \subset I^-(\mathcal{J})$ . Furthermore it is important to note that, contrary to a common misconception, the datum of  $U$  is an additional one, and in general there are null geodesics of infinite affine parameter that do not end in  $\mathcal{J}$ . Putting an appropriate conformal factor  $a$  on Kruskal spacetime yields a w.a.s. spacetime for which no choice of  $U$  and  $\mathcal{J}$  is possible such that all inextendible null geodesics of infinite affine parameter would end at  $\mathcal{J}$ : Just choose  $a = 1$  outside of a small neighborhood  $U$  of the left future null infinity and between  $1/2$  and  $3/2$  in  $U$  but oscillating sufficiently wildly towards the left future null infinity to prevent any conformal extension there (e.g., by violating Penrose's peeling estimate). Kerr spacetime is an example of a weakly asymptotically flat spacetime for which no choice of  $U$  and  $\mathcal{J}$  is possible such that all inextendible null geodesics would end at  $\mathcal{J}$ . Here the exterior region is the largest open set that can be chosen as  $U$ , and although there is a second open set of TIPs corresponding to the upper future null boundary in the second region of the usual diagram, no open set as above can cover it, as all null geodesics emanating from the second region have finite affine length.

A weakly asymptotically simple spacetime is called *asymptotically predictable from a partial Cauchy hypersurface*  $S$  iff  $\mathcal{J} \subset \text{cl}(D^+(\psi(S)))$ , where  $\psi$  is the Penrose extension appearing in the definition of asymptotic simplicity and thus in weak asymptotic simplicity. In Prop. 9.2.1 (p.311) of the book of Hawking-Ellis it is proven that if  $S$  contains an OTS  $A$ , then

$$(1) \quad J^+(A) \cap \mathcal{J} = \emptyset.$$

Theorem 16 implies that if we have a strong conformal extension  $F$  and if we define  $\mathcal{J}_{\max}(F)$  as the subset of all endpoints of null geodesics of infinite affine length (well-posedness of  $\mathcal{J}_{\max}(F)$  is shown below), then

$$(2) \quad J^+(A) \not\subset \mathcal{J}_{\max}(F).$$

and below we will prove that under the conditions of Theorem 16 and the extra condition of a strong conformal extension we can show that even

$$(3) \quad J^+(A) \cap \mathcal{J}_{\max}(F) = \emptyset.$$

Remark 19 implies that Condition 3 does *not* follow from the condition 1. Whereas condition 1 and condition 2 will not be enough to conclude the existence of black holes, the last condition is indeed sufficient.

As, for a conformal extension  $F : M \rightarrow \hat{M}$ , the target manifold  $\hat{M}$  is distinguishing, two curves  $c_1, c_2$  have the same endpoint  $q$  in  $\partial^+ F(M)$  if and only if their pasts coincide. We also know that for every null geodesic  $c$  in  $M$ , the curve  $F \circ c$  is a null pregeodesic in  $N$  and therefore intersects  $\partial^+ F(M)$  in a unique first point  $q_c$ . Let now  $F : (M, g) \rightarrow (\hat{M}, h)$  be a *strong* conformal extension, i.e.  $h = \omega \cdot (F_* g)$  for a function  $\omega \in C^\infty(F(M))$ , the pushforward of the inverse of the conformal factor, with a  $C^k$  extension to  $\hat{M}$ . We call  $q \in \partial^+ F(M)$  an *infinity point* if  $\omega$  is



$C^0$  extendible by 0 at  $q$  and a *finiteness point* if  $\omega$  is  $C^0$  extendible by  $a > 0$  at  $q$ . Obviously, the set of finiteness points is open in  $\partial^+ F(M)$ . For well-posedness of  $\mathcal{J}_{\max}(F)$  as above we show the following variant of a well-known fact:

**Theorem 20.** *Let  $(M, g)$  have a strong conformal extension  $F : M \rightarrow \hat{M}$ . Let  $c$  be a  $C^0$ -inextendible null geodesic in  $(M, g)$ . Then  $c$  is of finite affine length if and only if  $q_c$  is a finiteness point.*

*Proof.* One direction is easy: Obviously, if a  $C^0$ -inextendible null geodesic in  $(M, g)$  has *infinite* affine length then  $q_c$  is an infinity point. For the other direction, define  $k := F \circ c$ . It is a classical fact that  $k$  is a lightlike pregeodesic in  $\hat{M}$ , and the affine parameters  $s$  of  $c$  and  $\hat{s}$  of  $k$  are related by  $ds = \omega^{-2} d\hat{s}$ . Now assume that  $q_c$  is an infinity point, i.e.  $\omega(q_c) = 0$ . For simplicity parametrize  $k$  in the past sense, with  $k(0) = q_c$ . Then, as  $\omega \in C^1(\hat{M})$ , we have that  $u := \omega \circ k$  is  $C^1$  as well, and on any finite interval  $[0, T] \ni t$  we have  $u(t) \leq D \cdot t + C \cdot t^2 \leq Et$  for an appropriate  $E > 0$ . Thus the affine future length  $l$  of  $c$  can be calculated as  $l = \int_0^T u^{-2}(s) ds \geq \int_0^T E^{-2} s^{-2} ds = \infty$ , which implies the claim.  $\square$

There is a corresponding very elementary theorem for timelike maximal geodesics, without the assumption of a conformal extension:

**Theorem 21.** *Let  $c_1, c_2$  be two future timelike curves with  $c_2(0) \subset I^-(c_1) \subset I^-(c_2)$ . If  $c_1$  is infinite and  $c_2$  is maximal, then  $c_2$  is infinite as well.*

*Remark:* In particular, any two maximal curves with the same past have the same finiteness.

*Proof.* We parametrize  $c_1$  by arc length and assume  $l(c_2) = E < \infty$ . Choose  $n > 0$  such that  $c_2(0) \in I^-(c_1(n))$  and a timelike curve  $k$  from  $c_2(0)$  to  $c_1(n)$ . Let  $T > 0$  be such that  $c_1(n + 2E) \in I^-(c_2(T))$  and choose a timelike curve  $K$  from  $c_1(n + 2E)$  to  $c_2(T)$ . Then the curve  $K \circ c_1|_{[n, n+2E]} \circ k$  is a curve of length  $> 2E$  from  $c_2(0)$  to  $c_2(T)$ , in contradiction to maximality of  $c_2$ . Thus  $l(c_2) = \infty$ .  $\square$

Let us consider Kerr spacetime: Choose points  $q \in \text{Ext}(\text{Kerr})$  and  $x \in n_2 \subset \partial^+(\text{BH}(\text{Kerr}))$  as defined in Theorem 12, then there is no maximal timelike geodesic from  $q$  with endpoint  $x$ : If there were one, it would arise as a limit of maximal geodesics to  $c(n)$  where  $c$  is an incomplete geodesic with endpoint  $x$ , and those approximate  $i_+$ , in the limit ending there and never reaching  $x$ . Actually it would be contradictory to Theorem 21 if such a geodesic curve existed, as  $i_+$  is dominated by  $x$ . All curves ending in  $\text{BH}(\text{Kerr})$  have finite length, as there is an isometric continuation to Block III.

**Theorem 22.** *Let  $(M, g)$  have a conformal extension  $F : M \rightarrow \hat{M}$ . Then a precompact set  $U \subset M$  consists of visually compact points if and only if  $J^-(J^+(U))$  is spatially compact. For a compact subset  $R$  of  $LM$  let  $N_R$  be the union of images of maximal  $M$ -geodesics with initial values from  $R$ . If  $J^-(c(I))$  is spatially precompact for every maximal geodesic with  $\dot{c}(0) \in R$ , then  $J^-(N_R)$  is spatially precompact.*

*Proof.* We use compactness of  $J^+(\bar{U}) \cap \partial I(M)$  (by future compactness of  $\bar{I}(\bar{M})$ ) and show that the subset  $\partial_c^+ I(M)$  of points  $x$  on  $\partial^+ I(M)$  with  $J^-(x)$  spatially compact in  $I(M)$  by applying outer continuity of  $I^-$  at  $x$  in  $\hat{M}$  to the subset  $\partial C$  where  $C$  is a precompact open subset of a Cauchy surface  $S$  containing  $J^-(x) \cap S$ . The second statement follows similarly.  $\square$

Dirac-Higgs-Yang-Mills (DHYM) theories have a nice conformal behaviour: If  $(\psi, \phi, A)$  is a solution of DHYM theory on a g.h. manifold  $(M, g)$  then  $(\Omega^{3/2}\psi, \Omega\phi, A)$  is a solution of DHYM theory on  $(M, \Omega^{-2}g)$  (modulo the usual identification of spinors for two conformal metrics).

If we restrict to *constant* conformal factors  $\Omega(x) = c \in \mathbb{R}$ , i.e., pure rescaling, and to Maxwell theory, then the Einstein tensor is invariant under scaling of the metric, whereas the energy momentum tensor  $T$  of Maxwell theory, which reads

$$T(g, A)(e_i, e_j) = \frac{1}{4\pi} \left( \sum_a (\mathcal{F}_{ai} \mathcal{F}_j^a) - \frac{1}{4} \sum_{ab} (\mathcal{F}^{ab} \mathcal{F}_{ab}) g_{ij} \right),$$

has conformal weight  $-2$  (inverse of that of the metric), and is quadratic in the curvature tensor  $\mathcal{F} = dA$ , thus  $T(c^2 g, cA) = T(g, A)$ , so for every solution  $(M, g, A)$  of Einstein-Maxwell theory,  $(M, c^2 g, cA)$  is another solution.

*Proof of Theorem 4.* Let  $R$  be the subset in  $TW$  of outer null vectors  $w$  such that the  $M$ -inextendible geodesic  $c$  with  $c'(0) = w$  has empty intersection with  $I^+(W)$ . Those geodesics are maximal on their entire interval of definition and thus have finite affine length due to Theorem 14. It is easy to see that  $J^-(J^+(\text{int}(W)))$  is spatially precompact iff  $J^-(N_R)$  is. Assume that  $J^-(N_R)$  is not spatially precompact, then Theorem 22 implies that there is at least one such geodesic  $\gamma$  such that  $I_M^-(\gamma)$  is noncompact. Let us denote the future end point of  $F \circ \gamma$  in  $\overline{F(M)} \subset N$  by  $s$ , which is a finiteness point according to Theorem 20. As  $I^-(\gamma)$  is noncompact and the maximal solution has standard spatial infinity,  $J^-(s)$  contains an infinity point  $q$ , contradicting the following lemma:

**Lemma 23.** *Let  $s$  be a finiteness point in a strong conformal extension  $F : M \rightarrow N$  of a Maxwell-Einstein solution for Zipser-asymptotically flat initial values. Then for any  $r \in \partial J^-(s) \cap \partial^+ F(M)$ , there is no infinity point on  $J^+(r) \cap J^-(s)$ .*

*Proof.* Choose a past causal curve  $c : [0, 1] \rightarrow N$  from  $s$  to  $r$ . The image of  $c$  is contained in  $\partial F(M)$  because  $\overline{F(M)}$  is causally convex. Let  $r_+ = c(1 - \delta)$  be the first infinity point along the curve from  $s$ . Causal convexity implies that  $J^+(r) \cap J^-(s) \subset N \setminus F(M)$ . As  $\partial^+ F(M)$  is an achronal boundary,  $J^+(r) \cap J^-(s) \cap \overline{F(M)} = c([0, 1]) \subset \partial F(M)$ . Now we want to find a Cauchy surface  $S$  of  $N$  such that  $\omega$  is positive on  $J^+(r_+) \cap S$ . It is easy to show (using Theorem 3 from [21] as re-proven in our Theorem 7) that  $J^+(r_+) \cap J^-(S) = D^-(S)$  for any Cauchy surface  $S$ . By strongness of the extension,  $\nabla d\omega$  is positive definite in a neighborhood of  $J^+(i_0)$  in transversal directions, and  $d\omega = 0$  on  $J^+(i_0)$ , thus by considering the Taylor expansion we conclude that there is a neighborhood  $U$  of  $r_+$  s.t.  $U \cap J^+(r_+)$  does not contain infinity points, i.e.  $\omega$  is positive on  $H := J^+(r_+) \cap S$  for  $S$  appropriately chosen. This is easily seen by considering a covering of  $H$  by two subsets, one containing the point of intersection with  $c$ , where positivity is shown by finding past curves to  $c(1 - \epsilon)$ , and a complementary set where positivity is shown by examining past curves to  $c(1 - \epsilon) = r_+$ . As  $H$  is compact, we can even find an open neighborhood  $H'$  of  $H$  in  $S$  such that  $\omega$  is still positive on  $H'$ . Of course  $r_+ \in \tilde{N} := D^-(H')$ .

As, on the other hand, the operator  $B \mapsto \mathcal{F}(B)$  induces the linear symmetric-hyperbolic Dirac operator on forms in Lorenz gauge, in  $\tilde{N}$  we have a solution  $(h, \tilde{A})$  of the Einstein-Maxwell equation.

Consider the Maxwell-Einstein action  $L_{ME}$  restricted to the class of metrics of the form  $u \cdot h$ , that is, we ask for critical points of  $L_{ME}$  in the class  $C^k(\tilde{N}, (0, \infty)) \cdot h \times \Omega^1(\tilde{N})$ . The Euler-Lagrange equations are obtained by just taking the trace of the usual Einstein equations and use that  $T(uh, A)$  is trace-free:

$$0 = \text{tr}_{uh}(T(uh, A)) = \text{tr}_{uh}(\text{Ein}^{uh}) = \text{scal}^{uh} - \frac{n}{2} \text{scal}^{uh},$$

or equivalently,  $\text{scal}^{u \cdot h} = 0$ , and using the classical formulas of conformal transformations for curvature we get, with  $u = e^{2f}$ :

$$L(f, A) := (2(n-1)\square^h f - (n-2)(n-1)\|df\|_h^2 + \text{scal}^h, \square^h A) = 0,$$

which is a normally hyperbolic semilinear equation for  $(f, A)$ . We consider the operator  $\Phi$  given by

$$\Phi(a) := \text{Ein}(e^a \omega h) - T(h, \tilde{A}).$$

The semilinear symmetric-hyperbolic operator  $\Phi$  satisfies  $\Phi(0) = 0$ . Uniqueness of solutions of  $\Phi = 0$  for the induced initial values  $a_0 := \ln \Omega$  restricted to  $H'$  tells us that the solution  $a$  restricts to  $\ln \Omega$  in  $F(M) \cap \tilde{N}$  and is therefore not extendible to  $r_+$ . Thus one can use Theorem 5.6 (with  $v=0$ ) or Corollary 5.9 of [15], which are versions of the standard lifetime estimate for semilinear symmetric hyperbolic operators  $Q$  with  $Q(0) = 0$  and  $C^k$  coefficients, to rescale the bounded initial value  $a_0$  on  $\Sigma$  by a small constant factor  $b > 0$  until the solution to  $\Phi = 0$  with initial value  $b \cdot a_0$  exists in a region including  $r_+$ . On the other hand, uniqueness of solutions implies that the rescaled initial values give rise to the known solution  $e^b \Omega$  with the same maximal domain of definition as  $\Omega$ , in particular *not* extendible to  $r_+$ , contradiction.  $\square$

*Continuation of the proof of Theorem 4.* By the beginning of the proof and by Lemma 23,  $J^-(N_R)$  is spatially precompact, and we can apply Theorem 2 and the fact that Maxwell theory satisfies the dominant energy condition.  $\square$

Let us have another look at Kerr spacetime: There are null geodesics in the second region whose past contains the entire first region, which in turn contains timelike and null curves of infinite length. With Lemma 23, it follows that Kerr spacetime (remind that this comprises the first and the second region!) does not admit conformal extensions, in particular not the one suggested by its projection diagram. At a first sight, this seems to be in contradiction to Theorem 1. However, recall that the Cauchy hypersurfaces in Kerr are diffeomorphic to  $\mathbb{R}^3 \setminus \{0\}$ . In this sense, the initial values of the Kerr-Newman family are connected to the trivial ones on  $\mathbb{R}^n \setminus \{0\}$ , whereas we should be more interested in families of initial values connected to the trivial one on  $\mathbb{R}^3$ , where Zipser's result shows that there is indeed a neighborhood around zero admitting conformal extensions.

*Proof of Theorem 5.* Using the terminology of [32] we see that  $(\mu - \|J\|)_{cg} = \frac{1}{c^2}(\mu - \|J\|)_g$  and  $\text{Rad}^{cg}(U) = c \cdot \text{Rad}^g(U)$  for every open subset  $U$  of the initial hypersurface. Consequently, the SY functional

$$SY(g, A, U) := \text{Rad}^g(U) \cdot \sqrt{\min\{\mu_A(q) - \|J_A(q)\| : q \in U\}}$$

is invariant under scaling the metric alone by a factor  $c^2$ . Due to the scaling behavior of the Maxwell-Einstein equations we have to scale  $A$  with the factor  $c$ . As the energy-momentum tensor is homogeneous in  $A$ , for any  $(g, A)$ , there is  $E > 0$  such that for  $I_E := (E^2 \cdot g, E \cdot A, U)$ , we have  $SY(I_E) \geq \sqrt{\frac{3}{2}}\pi$ . By Schoen-Yau's result [32] it follows that then  $I_E$  contains an OTS. Using a result of Galloway [12] one can show by compactness of the interior of the OTS that it is strictly stable and thus allows for a small deformation giving rise to a trapped surface. Then Theorem 4 implies that there is a black hole in  $(M, h)$ .  $\square$

## 6. CONCLUSION: THE DIFFICULTY OF COSMIC WEATHER FORECAST, OR: THE LAMENTABLE FATE OF CASSANDRA OBSERVERS

There are many attempts to replace the global notions of 'mass' and 'event horizon' by quasilocal notions, i.e., those that only depend on the data in a compact subset and can be decided by a single observer (maybe even in finite time). For the following considerations, let us adopt the viewpoint of describing the dynamics of the universe in terms of classical field theories and treat observers as test particles. To get things more down to earth, let us give an analogy with the weather forecast: A notion like 'the set of places where it is going to rain at some future time' is perfectly well-defined from the purely logical point of view. However, for practical purposes we at least feel the need to complement it by a second notion that can be decided within finite time, like 'the set of places where it will rain tomorrow'. Even more, we want to have *predictive* criteria, i.e. methods able to detect whether an event will arrive *before it arrives*. If a weather forecast informs the receivers of a storm in the very moment of its arrival such that no precautions can be taken any more, then we would call it *practically useless*.

Now, of course, the statement of the Theorem 4 can be understood as a strong singularity theorem for the particular matter model of Maxwell theory. However, it can also be understood as some sort of censorship statement, as the singularities (incomplete  $C^0$ -inextendible causal curves) emanating from the outer trapped surface are not visible from future null infinity. Actually, it implies an even stranger form of censorship: *The singularity criterion itself is not visible from future null infinity*. This means, any 'Cassandra' observer able to see only the tiniest part of an outer trapped surface, consequently not agreeing on nonexistence of outer trapped surfaces assumed by other observers and trying to warn others from its effect by a minority report, is already caught in the region of finite lifetime, together with anybody who could get the warning message. Certainly, this 'brutal optimism' is some sort of cosmic censorship, but probably a more cruel one than usually expected. And it shows that the pragmatic power of predictivity of the Penrose singularity theorem for Einstein-Maxwell theory in the asymptotically flat case is very limited, actually its statement is practically useless in the sense of the weather forecast criterion above. Speaking of the Hoop Theorem itself, Theorem 5, the details of the Schoen-Yau analysis show that the outer trapped surface cannot be too far away from the region satisfying the Schoen-Yau concentration condition, and it is currently under investigation whether in our case of asymptotically flat Maxwell-Einstein theory it is possible that the region does not intersect the outer trapped surface. Any knowledge on the matter equations that could be of practical use would have to allow to predict (at least statistically) the presence and the location of an outer trapped surface *W by measurements of regions outside of  $J^+(W)$  alone*. This in particular excludes criteria that require the knowledge of the solution of the whole Cauchy problem for  $J^-(W)$ . A criterion of this kind does not seem to be known up to now — the Penrose inequality, e.g., would yield an estimate in the other direction only. Monotonicity of certain quasilocal mass quantities under geometric flows, like the monotonicity of the Hawking energy under the inverse mean curvature flow, could play a central role here. However, whether statements of this kind can lead to a criterion of practical use, that is, to a method such that the joint future of the region that has to be measured is not contained in the future of  $I$  and therefore in the black hole, remains to be seen.

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